A Series Transformation for Diaphragm-Type Discontinuities in Waveguide

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***Abstract*—In treating the problem of the inductive/capacitive diaphragm in waveguide with the integral equation or the moments method, it is often necessary to compute the matrix elements of the guide dyadic in terms of the aperture eigenmodes of the diaphragm. A transformation of the original series is presented that displays superior convergence properties of the numerical solution.**

In the dynamic solution of the problem of the inductive/capacitive diaphragm in rectangular waveguide by means of the variational method, the integral-equation method [1], [2], or the method of the moments [3]–[5], it is often necessary to compute the matrix elements of the waveguide Green's function in terms of the diaphragm eigenmodes, i.e.,

$$M_{mk} = \iint_{\text{apert}} \psi_m(x) K(x, x') \psi_K(x')\, dx dx'. \tag{1}$$

Here, ψ_m denotes the mth eigenmode of the iris (of aperture d), and the kernel K is

$$K(x, x') = \sum_{n>1}^{\infty} \Gamma_n \sin \frac{n\pi x}{w} \sin \frac{n\pi x'}{w}. \tag{2}$$

w is the broad/narrow dimension of the guide for the inductive/capacitive iris, respectively, Γ is the propagation constant, and the sum includes all the nonpropagating modes ($n>1$). In terms of the ψ's, the nth eigenmode of the guide can be written as

$$\sin \frac{n\pi x}{w} = \frac{2}{d} \sum_{m=1}^{\infty} P_{mn} \psi_m(x), \qquad P_{mn} = \frac{2m\pi}{d} \cdot \frac{\sin n\tau}{\left(\frac{m\pi}{d}\right)^2 - \left(\frac{n\pi}{w}\right)^2} \tag{3}$$

and

$$\tau = \frac{\pi(w-d)}{2w} \qquad \psi_m = \sin \frac{m\pi}{d}\left(x - \frac{w-d}{2}\right)$$

for a symmetrical iris. The asymmetrical case proceeds on analogous lines. Introducing (3) and (2) in (1), the matrix elements M_{mk} can be written as

$$\left(\frac{2w}{\pi d}\right)^2 mk \sum_{n=3,5,\cdots} \Gamma_n \frac{\sin^2 n\tau}{(n^2-a^2)(n^2-b^2)} = \frac{4w}{\pi d^2} mk\, S \tag{4}$$

where n is odd, $a = mw/d$, and $b = kw/d$.

The series S appearing in (4) is convergent, but only as fast as $1/n^3$ and, since proper uniform convergence occurs only after n has become larger than a *and* b, the situation will deteriorate with an increasing number of iris eigenmodes taken into account.

Poles occur whenever $m/n \approx w/d$ or $k/n \approx w/d$. A more advantageous way of computing (4) is illustrated in the following. For simplicity we take the iris to be inductive. The capacitive case follows on analogous lines. We can separate the quasi-static contribution from the dynamic contribution in (4) by writing

$$S = S_0 + S_D = \sum_n \frac{n \sin^2 n\tau}{(n^2-a^2)(n^2-b^2)} + \sum_n \frac{\left(\frac{w}{\pi}\Gamma_n - n\right)\sin^2 n\tau}{(n^2-a^2)(n^2-b^2)}. \tag{5}$$

Since

$$\frac{w}{\pi}\Gamma_n = n\left(1 - \frac{1}{2}\left(\frac{2w}{n\lambda}\right)^2 + 0(n^{-4})\right), \qquad (n \to \infty)$$

the series S_D, the difference between the dynamic and static contribution, converges very rapidly. The static series S_0 is odd in the summation variable n and, as such, does not lend itself to the application of standard summation techniques. We can, however, turn this into an even series thanks to the following device [2, pp. 582–583]. Letting u denote a real parameter >0, we write

$$S_0 = Z_0 + Z_D = \sum_n \frac{n \cdot \sin^2 n\tau \cdot \tanh nu}{(n^2-a^2)(n^2-b^2)} + \sum_n \frac{n \sin^2 n\tau}{(n^2-a^2)(n^2-b^2)}(1 - \tanh nu) \tag{6}$$

for u "not too small." Z_D converges very rapidly, while Z_0 is even in n and lends itself to summation by contour integration. We begin by writing

$$Z_0 = -\frac{1}{4}\left[\sum_{-\infty}^{+\infty} \frac{n \exp 2in\tau \tanh nu}{(n^2-a^2)(n^2-b^2)} - \sum_{-\infty}^{+\infty} \frac{n \tanh nu}{(n^2-a^2)(n^2-b^2)}\right] = -\frac{1}{4}[S_1 - S_2], \qquad (n \text{ odd}) \tag{7}$$

where, for brevity of notation, we temporarily included here the term $n=1$ in the summation.

We can now sum S_1 and S_2 by contour integration [6]. Poles of $f_1(z) = z \exp 2iz\tau \tanh uz/(z^2-d^2)(z^2-b^2)$ occur at $z = \pm a$, $\pm b$, and $\pm i(n\pi/2u)$. Therefore,

$$S_1 = -\frac{\pi}{2(a^2-b^2)}\left[\tanh au \frac{\sin a\left(2\tau - \frac{\pi}{2}\right)}{\cos \frac{\pi a}{2}} - \tanh bu \frac{\sin b\left(2\tau - \frac{\pi}{2}\right)}{\cos \frac{\pi b}{2}}\right] + \frac{\pi^2}{2u^2}\sum_{n=1}^{\infty} \cdot \frac{n \sinh \frac{n\pi}{u}\left(\tau - \frac{\pi}{4}\right)}{\left(\frac{\pi^2 n^2}{4u^2} + a^2\right)\left(\frac{\pi^2 n^2}{4u^2} + b^2\right)\cosh \frac{\pi^2 n}{4u}} \tag{8}$$

and S_2 is obtained by letting $\tau \to 0$ in (9).

Substituting in (7) and using (4)–(6) we finally obtain the required matrix element as

$$M_{mk} = \frac{4}{d^2} mk \frac{w}{\pi}[Z_D + S_D + Z_0]$$

$$= \frac{4}{d^2} mk \frac{w}{\pi}\left[\sum_{n=1}^{\infty} \frac{\left(\frac{w}{\pi}\Gamma_n - n\right) + (1 - \tanh nu)}{(n^2-a^1)\cdot(n^2-b^2)} \sin^2 n\tau - \frac{\pi^2}{8u}\sum_{n=1}^{\infty} \frac{2n \sinh \frac{n\pi\tau}{2u} \cosh \frac{n\pi}{2u}\left(\tau - \frac{\pi}{2}\right)}{\left(\frac{\pi^2 n^2}{4u^2} + a^2\right)\left(\frac{\pi^2 n^2}{4u^2} + b^2\right)\cosh \frac{\pi^2 n}{4u}}\right]. \tag{9}$$

In spite of its cumbersome look, (9) is computationally much more satisfactory than the original series (4). In fact, the resonant behavior is confined to the first series in (9), where the second series in (9), representing the static contribution, converges as fast as

$$\frac{1}{n^3} \exp\left[-\frac{n\pi}{u}\left(\frac{\pi}{2} - \tau\right)\right]$$

and it depends, therefore, upon the value of $(\pi/2) - \tau = (\pi/2)(d/w)$, i.e., the ratio of the aperture to waveguide dimensions. For given n and $\epsilon > 0$, the "optimum" value of u is determined from the condition $|M(n, u) - M(n+k, u)| < \epsilon$ (k arbitrary), which is tantamount to requiring uniform convergence. This, however, involves the solution of rather bulky transcendental equations in u. Since convergence after the first few terms is rather insensitive to the value of u in a relatively large range $0 < u < 1$, a more convenient method of ensuring uniform convergence was preferred. This consisted in taking for u the smallest value such that the term $(1 - \tanh nu) - (n - (w/\pi)\Gamma_n)$ in the (potentially resonant) first series of (9) is less than ϵ for a (approximately) given n. This method is obviously not optimal and other criteria could be given to yield a satisfactory estimate of u.

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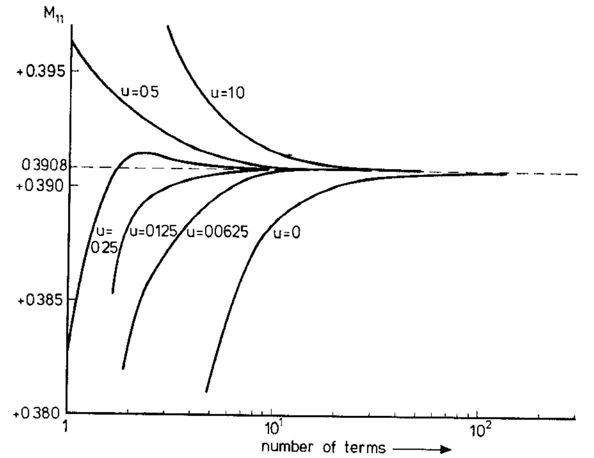

Fig. 1. Matrix element M_{11} as a function of number of terms in the series.

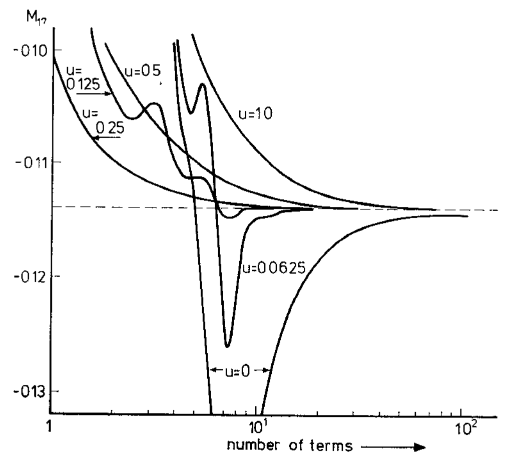

Fig. 2. Matrix element M_{17}.

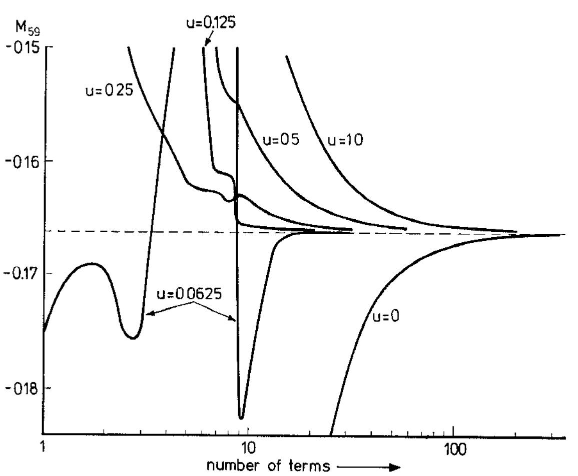

Fig. 3. Matrix element M_{59}.

As an example, the matrix element M_{11} was computed for the following values of the parameters: $d/w=0.4190$ and $d/\lambda=0.3431$ by means of (4) and (9). With these dimensions, the fundamental mode of the guide is above cutoff and its contribution does not appear in the series (4). This is tantamount to taking $\Gamma_1=0$ in (9).

The results are shown in Fig. 1, where the sum of the series truncated after terms $(N=(n+1)/2)$ is plotted against N for various values of u. The curve corresponding to $u=0$ is the original series (4), as can be seen by inspection of (5) and (6).

No resonance can occur in M_{11}. An example of its occurrence is illustrated in Fig. 2, where the result of truncating the series in the matrix element M_{17} after N terms is plotted against N. For a symmetrical aperture, $k=7$ corresponds to a four-term modal development. A resonance peak appears as n approaches 17, since then $k/n=7/17\simeq 0.419=d/w$ and its contribution to the sum is only slowly compensated by that of following terms of the opposite sign. It follows that no fewer than 200 terms will be needed in order to achieve a fourth decimal accuracy. The situation deteriorates for larger values of m and k. The same accuracy is achieved after 10 terms of the modified series, with the value u set equal to 1/5 by a simple estimate.

As $u\rightarrow 0$, a resonance begins to show also in the transformed series, as it should do, by continuity. The appearance of two resonances for m and k larger than unity is illustrated in Fig. 3 ($m=5$, $k=9$).

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Time-Delay Limits Set by Dispersion in Magnetostatic Delay Lines

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***Abstract*—Analysis and experiments show the extreme dispersion of magnetostatic delay lines. A suitable parameter to characterize the amount of dispersion present has been found to be the maximum output energy contained in a time interval equal to the input pulse duration. The time occurrence of this maximum value gives a convenient measure of the group delay. The pulse shape and energy content versus delay have been determined both theoretically and experimentally for axially magnetized circular rods. The results show that delays beyond two to three times the input pulse duration cannot be obtained with more than 50 percent of the output energy contained within the original pulse duration.**

I. Theory

The magnetostatic-wave group delay in axially magnetized ferrite rods has been expressed as [1]

$$\tau_g(\omega) = \frac{\alpha}{\omega - \omega_c} \tag{1}$$

where ω_c is the cutoff angular frequency of the volume modes and $\alpha = j_{mn}(1+q^2)^{5/4}/\sqrt{3}q^2$ [1] is a parameter taking care of the ferrite sample and the mode involved, q being the ratio of diameter to length of the rod and j_{mn} the nth root of the mth-order Bessel function. In our case we deal with the (1, 1) mode, $j_{01}=2.405$ [2]. Equation (1) assumes a parabolic internal field profile.[2]

To investigate the amount of dispersion introduced by the magnetostatic line, we want to derive the output signal corresponding to a rectangular pulse-modulated input frequency ω_0. To do this we assume a transfer function $A(\omega)\exp j\phi(\omega)$ with $A(\omega)=\text{const}=A$, and $\phi(\omega)$ approximated by a power expansion with the terms up to the

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[1] Our theory and experiments refer to two-port delay lines (input-to-output transmission) whereas [1] deals with one-port (pulse-echo) delay lines. The factor of 2 appearing in [1] is not, therefore, included here.

[2] For a more accurate computation, the Sommerfeld [3] field profile could be introduced. The improvement, however, did not show to be substantial for the commonly used aspect ratios, $q=0.2\cdots 0.3$ [4].